Deep Algebra Projects: Algebra 1 / Algebra 2
Aquarium Equations

Topics

- Linear functions (using tables, graphs, and formulas); slope and *y*-intercepts
- Solving linear equations
- Arithmetic and geometric sequences (brief experience with the latter)
- Iteration of functions; fixed points
- Mathematical modeling

In Aquarium Equations, students use a real-world situation involving salt concentrations in an aquarium as a springboard to explore the concept of *iteration* of mathematical functions. Iteration involves choosing an initial input and applying a mathematical function repeatedly with each output becoming the next input.

Stage 1 sets the scene by introducing the real-world context and asking students to carry out some preliminary investigations. The solution process to one of the problems involves iteration, but students do not know that yet!

In Stage 2, students take a break from the aquarium context in order to explore the mathematical aspects of iteration. The term *iteration* is defined, and students experiment with the effects of iterating a variety of linear functions using different initial inputs. In the process, they reinforce and develop their understanding of linear functions (using tables, graphs, and equations) and begin to explore geometric sequences.

Students return to the aquarium model in Stage 3 as they connect and apply what they learned in the first two stages in order to gain a deeper understanding of the long-term behavior of the aquarium's salt concentration. They finish by helping the aquarium owner create a plan to keep the salt levels in her aquarium at the best level for her fish and by reflecting on their mathematical model to consider its assumptions and get a sense for its limitations.

Stage 1

Distribute handouts for only the Stage 1 problems at this point. Introduce the scenario (the opener for Problem #1). Ask students to imagine what the problem(s) might be.

Discuss ways in which the problem situation is or is not realistic. If some of your students have experience with aquariums or would like to do some research on caring for fish in a saltwater aquarium, have them share their knowledge. Ask the class to keep these facts and practicalities in mind as they work on the exploration, especially if certain aspects of the problem situation do not seem realistic. After they have finished some of the mathematical analysis, they can return to these points and talk about how they may affect the interpretation of their results. Students may even come up with ways to improve the mathematical model!

Note: In Problem #3, you may need to take time to discuss the subscripts, n and $n+1$. Although some students may have seen subscripts before (such as in a formula for calculating slope), the symbols may appear confusing at first.

What students should know:

- Use fractions, decimals, and/or percentages to reason about mixtures.
- Use algebraic expressions to represent patterns in numbers and computations.
- Understand that linear functions are defined as having a constant rate of change.
- Understand slopes and y–intercepts of linear functions.
- Solve linear equations.

What students will learn:

- Apply knowledge of linear functions to model a real-world situation.
- Interpret mathematical results in a real-world context, and use them to make predictions or decisions.

Problem #1

Pam has a 32-gallon saltwater aquarium with 4250 grams of salt dissolved in the water. To keep her fish healthy, she is trying keep the salt content between about 4100 and 4400 grams. Each week, four gallons of water evaporate from the tank.

Directions

At the end of the week, Pam replaces the evaporated water with tap water containing 1.2 grams of salt per gallon.

- Calculate the amount of salt in the aquarium for each of the next five weeks.
- Determine if amount of salt in the aquarium is a linear function of the time. If not, explain why not. If so, find the rate of change (slope) and the starting value (y-intercept). Explain what will happen over the long term.
- Suggest a general strategy for Pam to keep the salt level in the desired range.

Conversation Starters for #1

What do you notice? What do you wonder?

I wonder what happens to the salt when the water evaporates.

The salt remains in the tank. Only the water evaporates!

I wonder if it is realistic for 4 gallons of water to evaporate each week.

I notice that the *concentration* (percentage) of salt in the water is probably more important than the total *amount* of salt.

It is the *concentration* that will affect the fish.

I wonder if 4100 to 4400 grams of salt per 32 gallons of water is a realistic concentration for saltwater.

I wonder if 1.2 grams of salt per gallon is a realistic concentration for tap water.

I notice that the concentration increases throughout the week as the water evaporates.

I wonder if the concentration of salt at the end of the week (before Pam refills the tank) is too high for the fish.

If this is true, then perhaps Pam should refill the tank more often. Later on, some students may want to explore this further and make any necessary adjustments to the problem situation.

I wonder if I can use a graph to check that the function is linear.

If students try this, they will have to recognize that the amount of change per week is very small relative to the amount of salt. This makes it more challenging to set up the scales in the coordinate system.

Solutions for #1

The salt levels over time

Since the salt does not evaporate with the water, the amount of salt in the aquarium will increase by 1.2 grams/gallon • 4 gallons = 4.8 grams each week. The amounts (in grams) for the first few weeks are:

4250, 4254.8, 4259.6, 4264.4, 4269.2, 4274, etc.

Deciding if the relationship is linear

The amount of salt is a linear function of time, because it increases at a constant rate of 4.8 grams per week. The starting value is 4250 grams.

After about 30 weeks, the salt levels will begin to get too high for the fish, so Pam will need to do something to get the levels back down.

How Pam could control the salt content

Students may suggest a variety of methods. The purpose of the question is just to get them thinking about the situation. In this activity, students will explore the effects of removing some of the salt water each week just before refilling with tap water.

Problem #2

Pam decides to control the salt levels by removing some water from the tank each week just before using the tap water to bring the level back to 32 gallons.

Directions

Suppose Pam removes 1 gallon of water each week just before refilling the tank.

- Make a rough prediction (without calculating) of what is likely to happen over time to the amount of salt in the water.
- Calculate the amount of salt in the tank each week for at least the first 5 weeks. Include a sample of your calculations and an explanation of your thinking.
- Is the amount of salt in the tank a linear function of time? Explain.
- Give Pam some general advice about what to try next. Explain.

Conversation Starters for #2

What do you notice? What do you wonder?

I notice that removing water from the tank affects the concentration more than adding an equal amount of tap water.

This makes sense, because the aquarium water is much saltier than the tap water.

I notice that it matters for the salt to be well mixed in the water.

If some parts of the water are saltier than others, it will not be possible to predict the effects of removing water from the tank.

I wonder if the salt is likely to stay well mixed without stirring.

It is probably reasonable to assume that the salt stays well mixed (i.e., that it is *uniformly distributed* throughout the tank).

I notice that there are 28 gallons of water in the aquarium at the point that Pam removes an additional gallon.

I notice that Pam must replace both the evaporated water and the water that she removed.

I notice that the salt levels decrease by a lot when Pam removes 1 gallon of water.

I wonder if the salt levels will keep decreasing until there is none left.

I notice that I keep doing the same calculation process for each week.

This makes sense, because the calculations describe the effects of what Pam is doing each week, and she keeps doing the same thing!

I wonder if my calculator has a fast way of doing these repeated calculations.

Many graphing calculators do. This will be discussed in the Conversation Starters for Problem #8. It is probably better for students to practice doing the calculations without this shortcut for a while first.

I wonder if removing less than 1 gallon of water per week will keep the salt level from changing too quickly.

Solutions for #2

Predicting the general effects of removing 1 gallon each week before refilling

Because the water that Pam is removing is much saltier than tap water, the salt level in the tank will drop. It may drop so quickly that it will become too low very soon.

Many students may predict that, if Pam keeps doing this, the salt level will drop until there is almost none left. Others may notice that the amount of salt will drop more and more slowly as time goes on, because as the water becomes less salty, removing it has less of an effect on the total amount.

The amount of salt in the tank during the first 5 weeks (nearest gram)

Week 0	Week 1	Week 2	Week 3	Week 4	Week 5
4250 g	4104 g	3964 g	3828 g	3697 g	3571 g

Sample calculations and an explanation

$$4250 \cdot \tfrac{27}{28} + 6 \approx 4104.2$$

$$4104.2 \cdot \tfrac{27}{28} + 6 \approx 3963.6$$

$$3963.6 \cdot \tfrac{27}{28} + 6 \approx 3828.0$$

After 4 gallons of water have evaporated, there are 28 gallons left in the aquarium. When Pam removes 1 gallon of the saltwater, she removes $\frac{1}{28}$ of the salt with it (assuming that the salt is uniformly distributed in the tank), leaving $\frac{27}{28}$ of the salt behind. The tank is now 5 gallons low on water. When Pam replaces the 5 gallons with tap water, she adds 1.2 grams per gallon • 5 gallons = 6 grams of salt.

Note: Since Pam does the same thing every week, the same calculation process (multiplying by $\frac{27}{28}$ and adding 6) applies each week. However, she begins with less salt each time. The table shows the results to the nearest gram, but it is important to keep many decimal places during the calculation process. Otherwise, small errors will accumulate week by week.

Linearity

The amount of salt is not a linear function of the time, because it is not decreasing at a constant rate: 4250 – 4104 = 146 g; 4104 – 3964 = 140 g; 3964 – 3828 = 136 g, etc.

Advice for Pam

The salt level is dropping much too quickly. Accept and acknowledge any reasonable advice that students offer. Some may suggest that she remove quite a bit less than 1 gallon water per week from the tank.

Problem #3

Before making more specific suggestions for Pam, it may help to think more about the mathematical ideas behind the calculation process.

Directions

- Write an equation that shows how to calculate the salt level (x_{n+1}) for week *n*+1 from the salt level (x_n) for week *n*. Explain your thinking.
- Determine whether the equation above is linear or not. If so, find the rate of change and *y*-intercept. If not, explain why not.

Conversation Starters for #3

What do you notice? What do you wonder?

I notice that writing the equation for the salt level for week *n*+1 just involves using variables to show my thinking process in Problem #2.

I wonder why the same letter, *x*, is used for two different variables.

The same letter is used because both variables stand for the same *kind* of thing—an amount of salt in the tank. They each have a different subscript because they refer to salt levels at different times.

I notice that the subscripts remind me of the formula for slope, because it has subscripts, too:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Again, the same letter is used for the two *y*-coordinates, but the 1 and the 2 in the subscripts keep track of which point each *y*-coordinate came from. (The same comment applies to *x*.)

I notice that *n* and *n*+1 do not refer to specific weeks, but that *n*+1 is always one week later than week *n*.

I wonder how this equation can be linear even though the salt level does not change in a linear way.

This equation only produces week by week before/after salt levels. It does not use time (the number of weeks) as the input.

I notice that the slope and *y*-intercept of the function do not describe the rate of change or starting value of the *amount of salt*, but they are still useful pieces of information.

Instead, the slope of this function represents the fraction of salt left after Pam removes 1 gallon, and the *y*-intercept stands for the amount of salt that Pam puts back into the aquarium when she replaces the missing tank water.

Solutions for #3

An equation for x_{n+1} in terms of x_n

Since the process of calculating one week's salt content from the previous week's salt content is the same every time, you can represent it with a single equation:

$$x_{n+1} = \frac{27}{28}x_n + 6$$

Depending on their previous experience, some students may need help (and time) understanding the subscripts n+1 and n. The same letter, x, is used for both variables, because they both represent the same kind of thing: the amount of salt. The subscripts are just labels that keep track of the week that the measurement refers to. Week n+1 is one week later than week n.

Note: In order to use this kind of formula to calculate the amount of salt in any given week, you must do the calculations for all of the previous weeks leading up to it! Formulas like this that involve computing each value in a sequence from one or more previous values are called *recursive*. It is quite challenging to find a formula that directly calculates the amount of salt in the nth week. Students will have a chance to try doing this in a Diving Deeper problem later in the exploration.

Deciding if the equation is linear

Even though the amount of salt is not a linear function of the time (see Problem #2), the week-by-week recursive equation above *is* linear! This shows (in a specific sense that students explore further in Stages 2 and 3) that the process of repeating a linear calculation over and over can give a non-linear result.

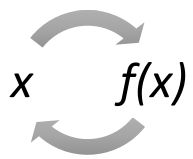
The constant rate of change (slope) of the equation above is $\frac{27}{28}$, and the y-intercept is 6. Notice that these numbers do not describe the rate of change and starting value of the amount of salt in the tank. They only describe how to calculate one week's amount from the previous week's amount.

Stage 2

Note: Do not hand out Stage 3 until students have completed the problems in Stage 2.

In Stage 2, students explore *iteration* (part of the process used in solving Problem #2). They will use relatively simple functions and numbers in order to better understand the underlying mathematics.

Before assigning the problems, you will need to discuss the concept of iteration. Its meaning is captured by the following type of diagram, which students will see in the upcoming problems.



Ask students to think about what the picture might mean. In order to iterate a function, choose a starting (or *initial*) input value for *x*, and find its output. (This is the rightward arrow.) Then, make this output the next input (the leftward arrow). Again, find the output, turn it into an input, and continue the process for as long as you like.

For example, suppose that you choose an initial input of 2 and the function $f(x) = x^2$. The iteration process will produce the sequence

2, 4, 16, 256, 65536, etc.

This is just like entering the number 2 on your calculator and pressing the "x^2" key over and over. You are choosing a number and squaring it repeatedly. In this case, the sequence grows extremely rapidly. However, if you choose a starting number of 0 or 1, it will not change at all. It will be a *constant* sequence of 0s or 1s. And if you choose a number between 0 and 1, the numbers in the sequence will get closer and closer to 0. The behavior of a sequence depends not only the function that you use but the input that you choose!

It helps to have a way to name the numbers in the sequences that iterations produce. You may choose any letter you like. We will use *x*. You might think that we would use *y* for the output, but since it gets turned back into an input, we stick with *x* all the way through. The initial input that you choose may be labeled with subscript of 0. The next uses a subscript of 1, etc. The numbers in the sequence are named like this:

$$x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots$$

Notice that beginning term is actually called the 0th term; the next one is the 1st term, etc. This may seem strange, but we are doing it this way in order to make some of our formulas easier to understand and work with.

As they work, students will discover that, even though they are using linear functions for *f*, their sequences will not be linear (*arithmetic*). In fact, they will encounter *geometric* sequences. If students are not familiar with them, you may describe them briefly when they arise and mention the word *geometric* if you choose. Geometric sequences are not the focus of the project, but students may benefit from experiencing them.

Students may or may not recognize it (and you may choose to let them discover it for themselves), but in Problem #2, they carried out an iteration process using an initial input of

$$x_0 = 4250$$

along with the function

$$f(x) = \frac{27}{28}x + 6.$$

In Stage 3, they will make this connection.

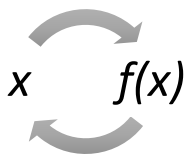
What students should know:

- Be familiar with the ideas from Stage 1.
- Know the meaning of a *function*, and use function notation, *f*(*x*).

What students will learn:

- Reinforce and extend concepts related to linear functions.
- Learn the meaning of *iteration*.
- Learn how to iterate functions.
- Represent and interpret the iteration process on a graph.
- Understand the concept of a *fixed point* of a function, and learn how to find it.
- Explore patterns in the iteration process, and explain what causes them.
- Distinguish the function that is used in the iteration process from the pattern that it produces in the sequence.
- Begin to explore *geometric* sequences.

Problem #4



The value of x that you start with is called x_0.

Directions

- Suppose that $f(x) = 2x$. Write the first ten numbers in each of the three sequences starting with: $x_0 = 0$, $x_0 = 1$, and $x_0 = 3$.
- A formula for the nth number in the $x_0 = 1$ sequence is $x_n = 2^n$. Find a formula for the nth number in the $x_0 = 3$ sequence.
- Suppose that $f(x) = \frac{1}{3}x$. Write the first seven number in each of the three sequences starting with: $x_0 = 0$, $x_0 = 1$, and $x_0 = 2$. Find formulas for the nth number in each sequence.
- Compare and contrast the sequences and formulas based on the two different functions.

Conversation Starters for #4

What do you notice? What do you wonder?

I notice that *f* is linear, but the sequences are not (except when x_0 is 0).

I notice that the sequences are *geometric*.

This means that neighboring terms have a *common ratio* (because each term results from multiplying the previous term by the same number).

I notice that the slope of *f* is the *common ratio* in the sequence.

I wonder if iterating a linear function always creates a geometric sequence.

I notice that changing the starting number just multiplied every term in the sequence by the same number.

I wonder why using $\frac{1}{3}$ in the function, *f*, is so different than using the number 2.

Remember that multiplying by numbers greater than 1 has a different effect than multiplying by numbers less than 1.

I wonder why the formulas for the sequences always seem to involve exponents.

Even the constant "0" sequence could be written using exponents. How?

I notice that the sequences are always increasing or always decreasing (when $x_0 \neq 0$).

I wonder if iterating a different kind of function could create a sequence whose terms increase at times and decrease at other times.

I wonder if changing x_0 always changes the sequence in an easily predictable way.

Solutions for #4

The first ten numbers for each sequence involving f(x) = 2x

Starting number $x_0 = 0$: 0, 0, 0, 0, 0, 0, 0, 0, 0, 0

(This is called a *constant sequence,* because its value never changes.)

Starting number $x_0 = 1$: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512

Starting number $x_0 = 3$: 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536

Note: Since the beginning number in each sequence corresponds to *n* = 0, we will call it the 0th number of the sequence. This may be a little confusing at first, but just remember to begin with 0 when you are counting to the position of a number in a sequence.

*A formula for the n*th number when $x_0 = 3$

$$x_n = 3 \cdot 2^n$$

The first seven numbers and formulas for each sequence involving f(x) = (1/3)x

Starting number $x_0 = 0$: 0, 0, 0, 0, 0, 0, 0 Formula: $x_n = 0$

Starting number $x_0 = 1$: $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}$ Formula: $x_n = \left(\frac{1}{3}\right)^n$

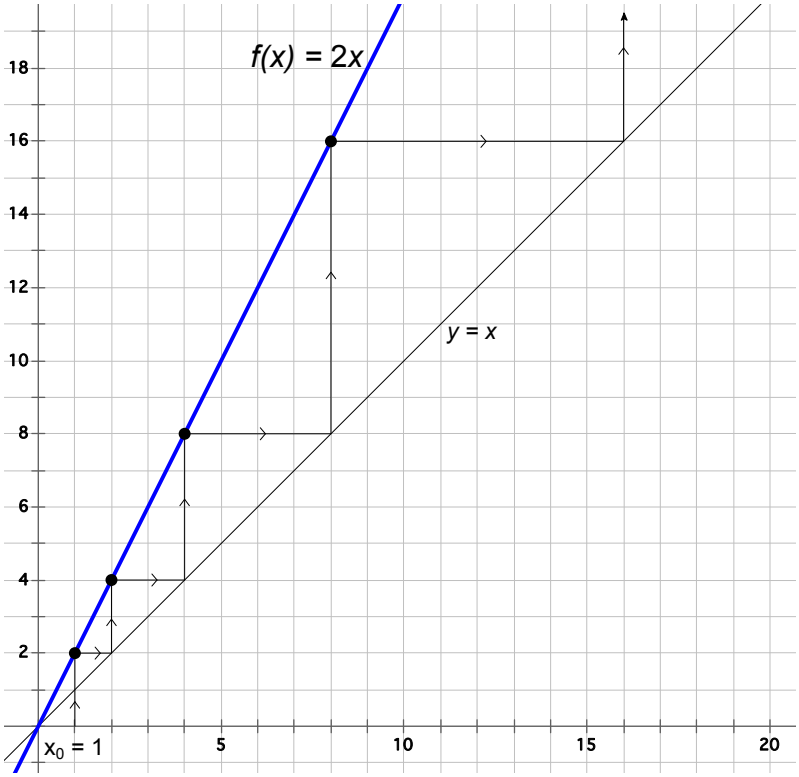
Starting number $x_0 = 2$: $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \frac{2}{243}, \frac{2}{729}$ Formula: $x_n = 2 \cdot \left(\frac{1}{3}\right)^n$

Comparing and contrasting the sequences and formulas

Both of the sequences that start with 0 are the same: repeated zeros. All of the sequences are *geometric*, meaning that they involve choosing a starting number and multiplying it repeatedly by some number.

Among the non-constant sequences, those based on *f(x) = 2x* keep growing forever (very quickly), because each number is twice the preceding number. The sequences based on *f(x) = (1/3)x* gradually shrink to 0, because each number is 1/3 of the previous number. In each case, the number that is repeatedly multiplied is the same as the slope of *f*. Thus, when the slope is greater than 1, the numbers in the sequence grow, and when the slope is less than 1 (between 0 and 1), the numbers shrink to 0.

Problem #5



Directions

- Study the graph carefully. Explain how it relates to Problem #4.
- Redraw the path in the graph using $x_0 = 3$ instead of $x_0 = 1$. Explain your thinking.

Diving Deeper

Create graphs like this for $f(x) = \sqrt{x}$ using a different starting number, x_0, for each graph. What do you notice? How do your graphs relate to the process of using the square root key on your calculator over and over?

Conversation Starters for #5

What do you notice? What do you wonder?

I wonder why the $y = x$ line is part of the drawing?

I notice that the path starts at x_0 and climbs in a staircase pattern.

I notice that the steps keep getting bigger and farther apart.

I notice that the edges of the steps bounce back and forth between the function, f, and the $y = x$ line.

I notice that the y-coordinates of the points where the steps touch f are exactly the numbers in the sequence produced by f!

I wonder if a $y = x$ line is always included in graphs that show function iterations.

I wonder why the $y = x$ line works to create this sequence.

I notice that the input is always equal to the output on the $y = x$ line.

I notice that the overall appearance of my graph is nearly identical after I change the starting number!

I wonder what would happen to the steps if the blue function line were closer to the $y = x$ line.

I wonder what would happen to the steps if the blue function line were below the $y = x$ line.

I wonder what would happen if the slope of the blue line were negative.

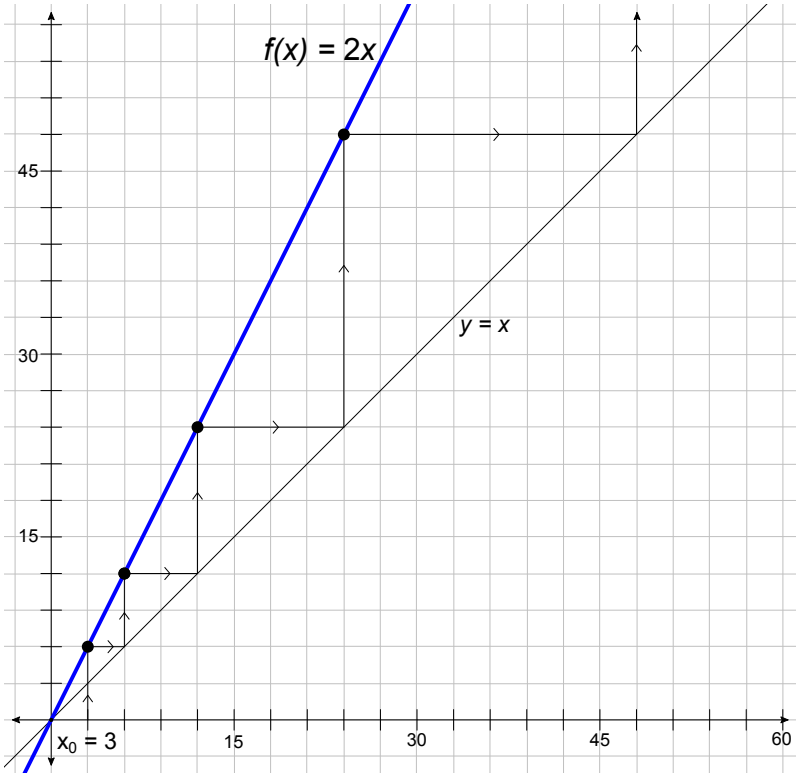
I wonder what would happen to the steps if the function with the blue graph were not linear.

Solutions for #5

How the graph relates to Problem #4

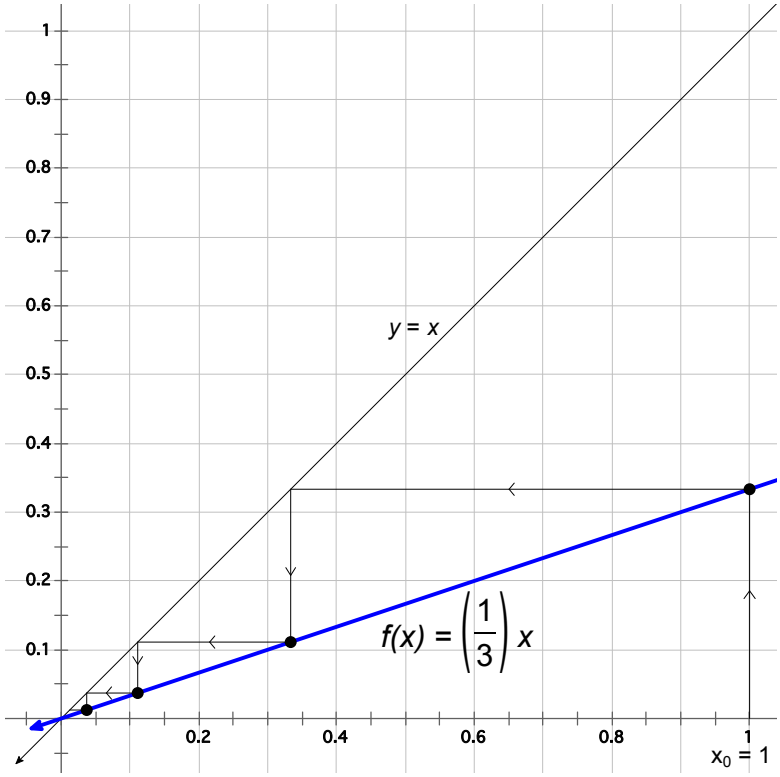
The path with the arrows shows the numbers in the $x_0 = 1$ sequence. Starting at x = 1, the path travels vertically to the $y = 2x$ line, reaching a point with a y-coordinate of 2, the next number in the sequence. From there it travels horizontally to the $y = x$ line* in order to find a point that also has an x-coordinate of 2. The output has become the new input! Now, by traveling vertically again to the $y = 2x$ line, the path reaches a point with a y-coordinate of 4, again the next number in the sequence. The process continues in the same way: each time the y-coordinate gives the next number in the sequence, you travel horizontally to the $y = x$ line in order to "turn the output into the next input." From there you again travel vertically to the $y = 2x$ line in order to find the next output. (*Note: The $y = x$ line is important because, on this line, the input equals the output.)

Redrawing the graph using $x_0 = 3$.



Some students may discover that they are essentially relabeling the axes, tripling each number on the scales! The new y-coordinates that are marked on the blue line now reflect the numbers in the $x_0 = 3$ sequence.

Problem #6



Directions

- Study the graph carefully. Explain how it relates to Problem #4.
- Compare and contrast the graph in this problem with the one in Problem #5.
- Predict the number of steps needed for the path to reach (0,0). Explain.
- Redraw the graph starting with $x_0 = 2$ instead of $x_0 = 1$. Explain your thinking.

Diving Deeper

What would happen if you began with a negative number for x_0?

Conversation Starters for #6

What do you notice? What do you wonder?

Most of the Conversation Starters from Problem #5 still apply to Problem #6.

I notice that the path now travels down and to the left.

I notice that the steps are now getting closer together.

I notice that the steps are crowding in toward (0,0)—the place where the lines meet.

I notice that the graph of f is now below the $y = x$ line.

I notice that I can change the starting number simply by changing the scales on the axes!

I notice that it is harder to read the y-coordinates of the points on the graph.

Imagine fractions that they appear to be close to.

Solutions for #6

How the graph relates to Problem #4

This graph shows the numbers in the $x_0 = 1$ sequence for $f(x) = \frac{1}{3}x$ in the same way that the graph in Problem #5 showed numbers in the $f(x) = 2x$ sequence. Traveling vertically to the graph of *f* produces the next number (output) in the sequence, and traveling horizontally to the *y* = *x* line changes this output into the next input.

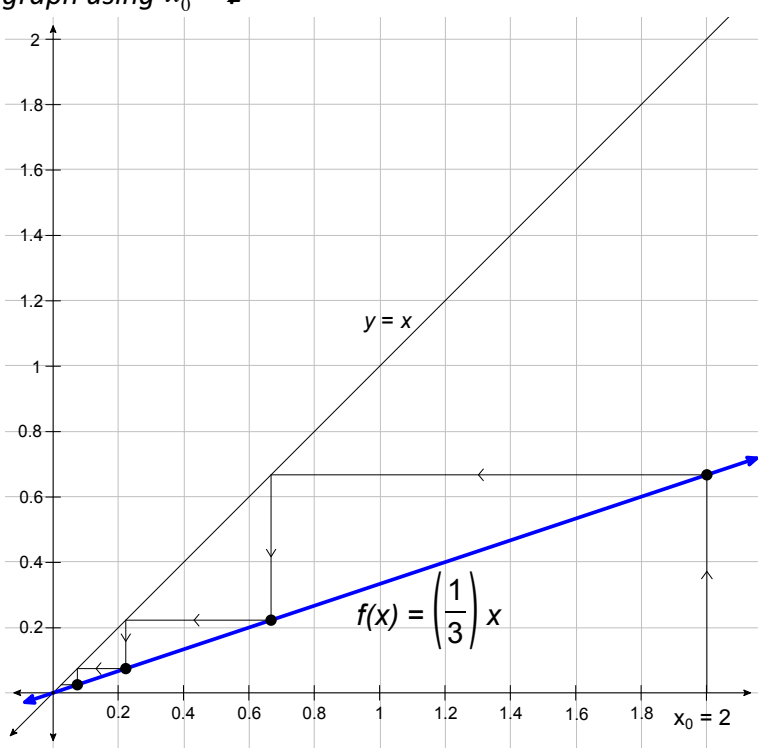
Comparing and contrasting with the graph in Problem #5

The graph in this problem is below the *y* = *x* line (at least in the first quadrant where we are looking). As a result, the path travels left and down instead of right and up, and the *y*-coordinates of the points on the graph of *f* are decreasing rather than increasing. This corresponds to the fact that the numbers in this sequence are decreasing.

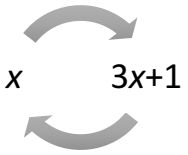
The number of steps needed for the path to reach (0,0)

The path will never reach (0,0). The *y*-coordinates on the blue line decrease as you continue along the path, but more and more slowly. This is reflected in the sequence, in which the numbers divide by 3 at each step, but never reach 0.

Redrawing the graph using $x_0 = 2$



Problem #7



Directions

- Write the first ten numbers in the sequence and create the graph using the starting number $x_0 = 1$. Describe what happens to the graph and sequence as they continue.
- Find a starting number that creates a *constant* sequence for the function $f(x) = 3x + 1$. (This starting number is called a *fixed point* for the function *f*.)
- If you have not already done so, explain how to use your graph and/or the equation $y = 3x + 1$ to locate this fixed point.

Diving Deeper

- Repeat Problem #7 choosing a negative integer for x_0. What changes and what stays the same? Why?
- Find a formula for the nth number in your sequence. (This is challenging! Teachers: See Looking Beyond on the final page of the project for the solution.)

Conversation Starters for #7

What do you notice? What do you wonder?

I wonder what effect the "plus 1" will have on the sequence.

I notice that the slope of f is greater than it was in Problem #5, which makes the graph of f farther away from the $y = x$ line.

I notice that the steps are even larger than in Problem #5 and that the sequence grows even more quickly.

I wonder if there is a connection between how far the graph is from the $y = x$ line and how fast the sequence grows.

I notice that the graph of f and the $y = x$ line no longer intersect at (0,0).

I wonder if there is a starting point that will make the path (and the sequence) get closer to this intersection point.

I notice that making a sequence constant still involves making the input of f equal its output.

I wonder why we didn't discuss fixed points in the previous problems.

They weren't very interesting, because the fixed point was always 0!

I wonder what happens if I choose a starting number very close to the fixed point.

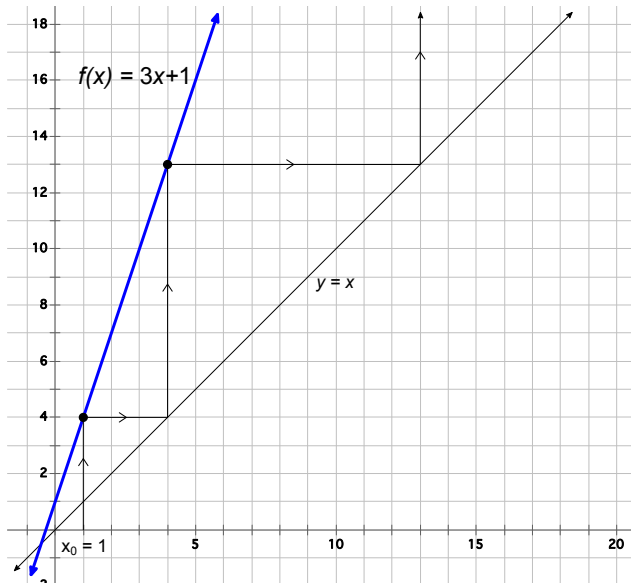
I wonder if there is a direct formula for this sequence, and if so, whether it involves powers of 3. (See the second Diving Deeper question.)

Solutions for #7

The first ten numbers the sequence

1, 4, 13, 40, 121, 364, 1093, 3280, 9841, 29524

The graph



The relationship between the sequence and the graph

The numbers in the sequence grow so quickly that only x_0, x_1, and x_2 fit on my graph. 1 is the starting point, and 4 and 13 are the *y*-coordinates of the large dots where the path meets the graph of *f*. The points belonging to later numbers in the sequence are far up and to the right. The numbers in the sequence grow more and more rapidly an *n* increases.

The starting number that creates a constant sequence

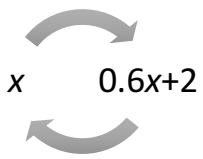
$x_0 = -\frac{1}{2}$ is the only number that creates a constant sequence, because it is the one number for which the input of *f* equals the output. Many students will probably use trial and error to discover this number.

Using the graph or table to find the fixed point

$(-\frac{1}{2}, -\frac{1}{2})$ is the point at which the graph of *f* intersects the *y = x* line. To find the point algebraically, students may write the equation, input = output:

$$x = f(x)$$
$$x = 3x + 1$$
$$-2x = 1$$
$$x = -1/2$$

Problem #8



Directions

- Write the first ten numbers in the sequence, create the graph, and find a fixed point for the function $f(x) = 0.6x + 2$ starting at $x_0 = 1$. Describe what will happen to the sequence as it continues. Explain your thinking.
- Explain how the linear function *f* can help you predict which sequences keep growing and which gradually approach a number.

Diving Deeper

- What happens if you choose a starting number greater than 5?
- Find a formula for the nth number in your sequence. (This is very challenging! Teachers: See Looking Beyond on the final page of the project for the solution.)

Conversation Starters for #8

What do you notice? What do you wonder?

Many of the Conversation Starters from Problem #7 still apply to Problem #8 (sometimes with small changes).

I notice that the steps keep getting smaller even though the graph of f is above the $y = x$ line.

I notice that the slope of f is less than 1.

I notice that the steps still approach the intersection point, but now this point is to the right of the starting number.

I wonder if there is a faster way to calculate the numbers in this sequence.

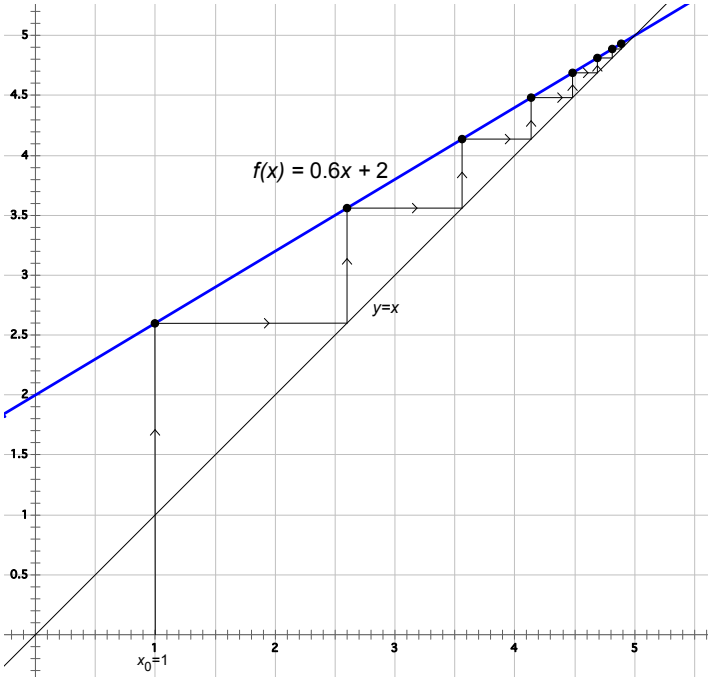
Some graphing calculators will do this.

- Type the starting number and press "ENTER."
- Type in the function, f, using the answer key (Ans) in place of x: 0.6Ans+2.
- Every time you press "ENTER," it gives you the next number in the sequence.

Solutions for #8

The sequence, graph, and fixed point

1, 2.6, 3.56, 4.136, 4.4816, 4.68896, 4.813376, 4.8880256, 4.93281536, 4.959689216



The fixed point is 5. (The graphs intersect at (5,5), and 5 is the solution of the equation $x = 0.6x + 2$.)

How the sequence will continue

Both the numbers in the sequence and the *y*-coordinates of the marked points on the graph appear to be increasing more and more slowly and getting closer to the fixed point, 5.

Using function, f, to predict to long-term behavior a sequence

Since the slope of the function is the multiplier in the sequence, slopes greater than 1 lead to sequences that keep growing. On the other hand, slopes between 0 and 1 cause the numbers in the sequence to change more and more slowly so that they gradually approach a number. Based on the examples in this exploration, it appears that the fixed point is the number that a sequence approaches. (Some students might be interested in exploring what happens when the slope of *f* is negative.)

Stage 3

In Stage 3, students integrate their learning from Stages 1 and 2. They apply their new mathematical understanding of iteration to the aquarium situation. They should always feel free to raise questions or doubts about the mathematical model as they work! This is an important part of the modeling process. Make note of their questions and return to them near the end of the project. In Problem #11, students are asked specifically to reflect on the usefulness and possible limitations of the model.

Notice that Problem #11 has no conversation starters. The "noticing and wondering" is built directly into the questions themselves.

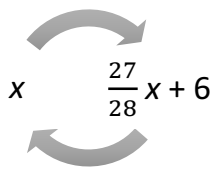
What students should know:

- Understand the problem situation and the concepts from Stages 1 and 2.

What students will learn:

- Apply knowledge of iteration to interpret real-world situations.
- Apply knowledge iteration to solve complex problems.
- Evaluate the usefulness and limitations of mathematical a model.

Problem #9



$x_0 = 4250$

Directions

- Explain how the diagram connects the ideas in Stages 1 and 2.
- Find the fixed point for $f(x) = \frac{27}{28}x + 6$. Show or explain your thinking.
- Explain what the fixed point tells you about the aquarium situation.
- What do you think will happen to the amount of salt in the aquarium over a very long period of time? Explain your thinking, and be as specific as you can.

Diving Deeper

Find a direct formula to calculate the amount of salt in the aquarium in week *n*.
(Teachers: See Looking Beyond on the final page of the project for the solution.)

Conversation Starters for #9

What do you notice? What do you wonder?

I notice that the formula for f is the same one that I used to calculate the amount of salt in the aquarium.

I notice that the starting number is the same as the starting amount of salt.

I notice that it makes sense for the output to turn into the input when I calculate the salt levels.

Yes, because the amount of salt at the end of the week is equal to the amount of salt at the beginning of the next week.

I wonder if I could draw a graph for this function and starting number.

Try it! It is possible, but it may be hard to draw and read. In order to show the large starting number on the graph, you need a scale that goes into the thousands. This makes the graph of f and the $y = x$ line appear to be very close together, which makes the steps hard to draw and see.

I notice that the numbers in the sequence still approach the fixed point, but they take a very long time to get there!

You may need to use a calculator shortcut like to one described at the end of Conversation Starters for Problem #8 in order to see this.

I notice that the slope of f is very close to 1.

I wonder if this is the reason that the sequence takes so long to get near the fixed point.

I wonder how small variations in the amount of water that Pam removes each week might affect the pattern of the salt levels approaching a fixed point.

Solutions for #9

How the diagram connects Stages 1 and 2

The starting number, 4250, and expression for *f* are the same as in the calculations for the salt levels in Problem #2. The sequence that you get from carrying out the iteration process (turning outputs bank into inputs and then calculating again) from Stage 2, represents the amount of salt in the tank each week.

It makes sense that the iteration process gives the salt levels, because the solution process in Problem #2 *is* an iteration! You start with a certain amount of salt and do the same calculation every week. The previous week's amount of salt (output) turns into the next week's beginning amount (input).

Many real-world phenomena can be modeled using iterations.

A fixed point for $f(x) = \frac{27}{28}x + 6$

The fixed point is 168. You can find this number by solving the input = output equation $x = \frac{27}{28}x + 6$ for *x*. You could also find it by sketching the graphs of *f(x)* and the line *y = x* and finding their point of intersection, but it may be tricky to choose useful scales on the axes.

What the fixed point tells you about the aquarium

If the aquarium had 168 grams of salt at the beginning (and Pam used the same process of removing 1 gallon each week and refilling with tap water), the amount of salt would stay the same every week.

The long-term behavior of the salt levels in the aquarium

The salt level will begin at 4250 and decrease steadily but more and more slowly. Over a *very* long period of time (It will still be 261 grams after two years!), it will gradually approach (and never become less than) 168 grams. Obviously, this is not an acceptable scenario. By week 2, the salt level is already too low! Pam needs to adjust her methods.

Problem #10

Pam wants to keep the amount of salt in her aquarium between 4100 and 4400 grams.

Directions

- Decide how much saltwater Pam should remove from her aquarium each week before refilling it with tap water. Explain your thinking.
- Explore and describe the behavior of the salt levels in the aquarium when Pam removes this new amount of saltwater each week. Explain your thinking.

Diving Deeper

- Explore the iteration process with some non-linear functions and with a variety of starting values. (Absolute value functions are an excellent choice.)
- Find a direct formula to calculate the salt levels in the aquarium when Pam removes this new amount of water each week. (Teachers: See the Looking Beyond section for the solution.)

Conversation Starters for #10

What do you notice? What do you wonder?

I notice that Pam should probably remove much less than 1 gallon of water each week, because the salt level dropped much too quickly in Problem #9.

I notice that the goal is to keep the *fixed point* between 4100 and 4400.

I wonder if there is a simple relationship between the amount of water that Pam removes and the fixed point.

It is not an especially simple relationship (which suggests that there is not a particularly simple strategy for calculating the amount of water that Pam should remove each week). See the note at the end of the Solutions.

I notice that the values *m* and *b* determine the fixed point of the iteration.

I notice that there is a predictable connection between the amount of water that Pam removes and the values of *m* and *b* in the equation for *f.*

Solutions for #10

The amount of saltwater that Pam should remove each week

Pam should remove about a half cup of saltwater each week.

Most students are likely to use a trial and error process in which they try to make the amount of salt removed equal to the amount added. This is very reasonable approach. (Their first guess should probably be much less than one gallon.)

Others may try an algebraic approach. The method below builds on the concept of a fixed point. (Keep in mind that *many* approaches are possible!)

The slope (*m*) and *y*-intercept (*b*) of the linear function, *f,* that produces a fixed point of 4250 are both related to the number of gallons that Pam removes, so it is worthwhile to work with these two values. You might start by realizing that *m* and *b* satisfy the fixed point "input equals output" equation:

$$x = f(x)$$
$$x = mx + b,$$

where *x* represents the fixed point. This determines a relationship between *m* and *b*:

$$x - mx = b$$
$$x(1-m) = b$$

Since the goal is for the fixed point to be near 4250, you can write

$$4250(1-m) = b.$$

This relationship between *b* and *m* is important, but it is not enough by itself to pin down their individual values or to determine the number of gallons (*g*) that you need to remove. Now is a good time to write down how each of *b* and *m* relate to *g*.

$$m = \frac{28-g}{28} \qquad b = 1.2(4+g)$$

It may take some thought to understand these equations, but they both involve exactly the same kind of thinking that students used to find the original function, *f*. *m* represents the fraction of salt that remains after *g* gallons are removed, and *b* stands for the amount of salt that is added from the tap water. (Notice that when $g = 1$, the equations give the correct values of $m = \frac{27}{28}$ and $b = 6$.)

You can now substitute these two expressions into the one above so that *g* is the only unknown!

$$4250\left(1-\frac{28-g}{28}\right)=1.2(4+g)$$

There are many ways to solve this for *g*. One approach begins by multiplying both sides of the equation by 28.

$$\begin{aligned}
&4250(28-(28-g))=33.6(4+g)\\
&4250g=33.6g+134.4\\
&4250g-33.6g=134.4\\
&4216.4g=134.4\\
&g=\frac{134.4}{4216.4}\approx 0.0319
\end{aligned}$$

You can multiply by 16 to express the number of gallons as cups (*c*). This gives

$c \approx 0.51$ cups.

To get a sense of the range of acceptable values, you may carry out the same process for *x* = 4100 and *x* = 4400 (the minimum and maximum acceptable levels of salt in the tank), resulting in approximately 0.49 to 0.53 cups. Thus, half a cup is in the acceptable range.

Note: In the process of finding *g* for these different values of *x,* some students may discover the general relationship between *g* and *x*. The relationship is described by the formula

$$g=\frac{134.4}{x-33.6}$$

where *g* is the number of gallons removed each week, and *x* is the fixed point (in grams of salt). This discovery is likely to come after they find the particular value of *g* for *x* = 4250—especially if they analyze their process for solving the equation at the top of this page and/or they start noticing patterns in their calculations for *x* = 4100 and *x* = 4400.

A great challenge would be to turn this equation into a very general formula that works for different rates of evaporation, different concentrations of salt in tap water, and aquarium tanks of any size! (Change some of the numbers above into variables as needed. Then redo the algebraic calculations.)

The new behavior of the salt levels

If Pam removes 0.5 cups of water each week, the fixed point will not be exactly 4250, but it will be very close. In fact, it will be slightly greater, because Pam is removing slightly less than the 0.51 cups suggested by her calculations.

Thinking process

Half a cup is 1/32 (or 0.03125) of a gallon. Since 32 • 28 = 896, this is 1/896 of the water in the tank when she removes it. Thus, 895/896 of the salt remains in the aquarium after Pam has removed half a cup of saltwater. Pam will need to replace 4.03125 gallons with tap water, which contains

1.2 grams per gallon • 4.03125 gallons = 4.8375 grams of salt.

The function that calculates each week's amount of salt is therefore

$$x_{n+1} = \frac{895}{896} x_n + 4.8375.$$

The fixed point for this function is 4334.4 grams of salt, which is a little greater than 4250, but still in the desired range as expected. Since the slope of this linear function is very close to 1, the salt levels will change *extremely* slowly as they approach this fixed point—probably too slowly to measure. In fact, slight variations in the amount of water that Pam actually removes and refills each week combined with other natural factors that may affect salt levels will almost certainly outweigh any changes predicted by this model. For all practical purposes, the salt levels will remain essentially constant, especially if Pam measures the salt concentration regularly and makes any necessary small adjustments.

Problem #11

The real world can be very complex. You must often make simplifying assumptions when you develop a mathematical model.

Directions

- Identify and describe some of the simplifying assumptions that went into creating the mathematical model in this exploration. Explain how these assumptions may affect the usefulness of the model.
- Most people are able to manage the environment in their aquariums without doing intensive calculations. Do you think that the mathematical analysis you have been doing is worthwhile? If so, for whom and for what purpose? If not, why not?

Solutions for #11

Simplifying assumptions in the model

Many assumptions went into building this model. Some may be fairly realistic, but many may not. We assumed that:

- The water evaporates at a constant rate.
- Nothing else is affecting the salt levels. (For example, other factors in the aquarium environment may require actions that affect salt levels.)
- Pam measures water with great accuracy.
- The salt levels in the tap water are consistent.

Students may have more ideas, especially if they have experience with aquariums.

These and other assumptions may result in salt measurements that are different than the ones predicted by the model. Hopefully, the model gives a reasonable idea of the amount of water to remove each week, but you are unlikely to see clear evidence of details such as a gradual approach to the fixed point, especially when the predicted changes in salt levels become small. Obviously, it will be important to measure the salt concentration regularly and to make adjustments as needed.

Note: There may other practical factors that affect the way that Pam cares for her fish. Some of these may require using an entirely different model.

- Tap water may not be the best choice for use in her aquarium.
- The fish may need to have the salt levels adjusted only gradually.
- It may not be advisable to wait an entire week before replacing evaporated water. In our problem scenario, the concentration of salt (due to evaporation) at the end of each week is probably too high for the fish.

The usefulness of the mathematical process

All reasonable answers should be accepted, but students should provide strong justifications for their conclusions.

Many students may believe that it is not worthwhile to do all of this math, since Pam could just buy measuring tools to monitor the salt levels and adjust as necessary. It is certainly true that most aquarium owners are unlikely to do these calculations! However, the model shows much more than specific numbers for Pam's situation. It offers insight into the *general behavior* of the salt levels under different circumstances—in particular, it shows the levels tend to approach a fixed point (after which they settle down) and that these fixed points and the rate at which they are approached may vary greatly. And, of course, in order to design aquarium ecosystems and provide advice to owners, *someone* must do this kind of math! In fact, the mathematical process of iteration behind this model provides deep insights into the behavior of countless natural and human-made systems.

Looking Beyond

Problem #7

Diving Deeper: a direct formula for the nth number in the sequence

$$x_n = \frac{3}{2} \cdot 3^n - \frac{1}{2}$$

or

$$x_n = \frac{3^{n+1} - 1}{2}$$

Problem #8

Diving Deeper: a direct formula for the nth number in the sequence

$$x_n = -4(0.6^n) + 5$$

or

$$x_n = 5 - 4(0.6^n)$$

A general pattern for the formulas

Some students may discover a general pattern for sequences generated by a linear function, *f*:

$$x_n = (x_0 - p)m^n + p$$

where x_0 is the initial input, *p* is the fixed point, and *m* is the slope of *f*.

Problem #9

You can use this general formula to find a direct formula for the amount of salt in Pam's aquarium in week *n*!

$$x_n = 4082\left(\frac{27}{28}\right)^n + 168$$

Remember to start counting the terms of the sequence at 0.

Problem #10

Diving Deeper: a direct formula for the amount of salt after *n* weeks when Pam removes only 0.5 cups of saltwater per week

$$x_n = -84.4\left(\frac{895}{896}\right)^n + 4334.4$$

or

$$x_n = 4334.4 - 84.4\left(\frac{895}{896}\right)^n$$

Note

Many sequences produced by iterating functions do not have direct formulas! Sequences produced by iterating *linear* functions are special in this sense.